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Extension of the q-deformed oscillator and two-parameter coherent states

M Arık and C Kılıç

Physics Department, Boğaziçi University, Bebek, Istanbul, Turkey

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Abstract
The $SU_q(2)$ algebra is extended by introducing additional raising and lowering operators and constructing their coherent states. This new algebra of coherent states and the commutation relations between the extended operators are investigated and a resolution of unity is proposed.

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1. Introduction

The q-deformed harmonic oscillator is a useful tool for quantum field theory since it constitutes a structure more compatible with interactions. The number q, viewed as a convergence parameter can be used to regulate divergences appearing in field theory calculations. Therefore, the deformed oscillator has been an active research topic over many years and several different representations have been introduced [1–4]. The starting point is to introduce a deformation of the commutation relation between the raising and lowering operators of the harmonic oscillator. In $SU_q(2)$ normalization [5]

$$aa^* - q^2 a^* a = 1 - q^2 \qquad 0 < q < 1. \tag{1}$$

There are other ways also of deforming the commutator [6, 7] which can be brought into this form by transformations of the operators. Equation (1) is the version that we will be using for our purposes. This algebra has one discrete and one continuous spectrum. The representation of the discrete spectrum is given by the set of basis vectors $|n\rangle$ such that

$$a^* |n\rangle = \sqrt{[n+1]}\,|n+1\rangle \tag{2}$$

$$a |n\rangle = \sqrt{[n]}\,|n-1\rangle \tag{3}$$

with

$$[n] = 1 - q^{2n} \tag{4}$$

and the ground state

$$a|0\rangle = 0 \tag{5}$$

so the states $|n\rangle$ are normalized eigenstates of the number operator a^*a.

$$\langle n_1|n_2\rangle = \delta_{n_1 n_2} \tag{6}$$

$$a^*a\,|n\rangle = [n]\,|n\rangle \tag{7}$$

with

$$|n\rangle = \frac{(a^*)^n}{\sqrt{f_n}}\,|0\rangle \tag{8}$$

and

$$f_n = [1]\,[2]\cdots[n]\,. \tag{9}$$

Coherent states are introduced as eigenstates of the ladder operators in the manner

$$|z\rangle = \sum_{n=0}^{\infty}\frac{z^n(a^*)^n}{f_n}\,|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{f_n}}\,|n\rangle \qquad |z| < 1 \tag{10}$$

such that

$$a\,|z\rangle = z\,|z\rangle \tag{11}$$

with the scalar product

$$\langle z_1|z_2\rangle = G^{-1}(\bar{z}_1 z_2) \tag{12}$$

where

$$G^{-1}(x) = \sum_{n=0}^{\infty}\frac{x^n}{f_n}. \tag{13}$$

This is easily verified using (10) and (6). Another important relation for G(x) is [8]

$$G(x) = \prod_{n=0}^{\infty}(1 - q^{2n}x) \tag{14}$$

from which it is easy to establish the formula

$$G(q^{2n+2}) = \frac{G(q^2)}{f_n} = \frac{f_\infty}{f_n}. \tag{15}$$

Using the Jackson integral, the resolution of unity is introduced as an integral over coherent states lying on circles with radii q^{2n} as

$$I = \sum_{n=0}^{\infty}|n\rangle\,\langle n| = \frac{1}{\pi(1-q^2)}\int_{|z|<1} G(q^2\,|z|^2)\;|z\rangle\,\langle z|\;\mathrm{d}_{q^2}z \tag{16}$$

where

$$\mathrm{d}_{q^2}z = \frac{1}{2}\mathrm{d}_{q^2}r^2\,\mathrm{d}\theta \tag{17}$$

and the Jackson integral is defined as

$$\int_0^a f(x)\,\mathrm{d}_q x = a(1-q)\sum_{k=0}^{\infty}q^k f(q^k a). \tag{18}$$

The normalization is chosen in order to ensure that

$$\langle m|\,I\,|n\rangle = \delta_{mn} = \frac{1}{\pi(1-q^2)}\int_{|z|<1} G(q^2\,|z|^2)\;\langle m|z\rangle\,\langle z|n\rangle\;\mathrm{d}_{q^2}z \tag{19}$$

where to evaluate the integral one uses

$$\int_0^1 x^n G(q^2 x)\,\mathrm{d}_{q^2}x = (1-q^2)f_n \tag{20}$$

which can be derived using (15).

2. $SU_q(2)$ algebra

Now let us review the $SU_q(2)$ formalism. It is worth remarking that the quantum group approach has several advantages. The deformation of the oscillator is identical to the q-deformation of the two-dimensional matrix representation of $SU(2)$. q, viewed as a convergence parameter, is useful in handling divergences appearing in interacting field theories. As mentioned, there exists another spectrum for the deformed algebra which is not discrete. In $SU_q(2)$ the identity is written as a sum of two positive definite terms which serves to discard this continuous part of the spectrum. This is done by adding two new operators. This constitutes the $SU_q(2)$ algebra with the following commutation relations:

$$ab = qba \tag{21}$$

$$ab^* = qb^*a \tag{22}$$

$$b^*b = bb^* \tag{23}$$

$$I = aa^* + bb^* = a^*a + q^{-2}b^*b. \tag{24}$$

These specify the action of the new operators on the basis states as

$$b\,|n\rangle = q^{n+1}\mathrm{e}^{\mathrm{i}x}\,|n\rangle \tag{25}$$

$$b^*\,|n\rangle = q^{n+1}\mathrm{e}^{-\mathrm{i}x}\,|n\rangle \tag{26}$$

where x is an arbitrary real parameter at this point. The action of the operators in the $SU_q(2)$ algebra on the coherent states is

$$a\,|z\rangle = z\,|z\rangle \tag{27}$$

$$a^*\,|z\rangle = z^{-1}(|z\rangle - |q^2z\rangle) \tag{28}$$

$$b\,|z\rangle = q\mathrm{e}^{\mathrm{i}x}\,|qz\rangle \tag{29}$$

$$b^*\,|z\rangle = q\mathrm{e}^{-\mathrm{i}x}\,|qz\rangle \tag{30}$$

$$a^*a\,|z\rangle = |z\rangle - |q^2z\rangle \tag{31}$$

$$b^*b\,|z\rangle = q^2|q^2z\rangle. \tag{32}$$

To make the phase appearing in the action of the operators b and b^* well-defined, we will introduce two-parameter basis states and extend the algebra somewhat further.

3. Extension of the algebra to two-parameter states

Now the action of the operators of the $SU_q(2)$ algebra on the coherent states suggests that we can introduce two-parameter coherent states $|z, w\rangle$ to well-define the arbitrary phase in the action of b and b^*. To do this we first consider states $|n, w\rangle$ where $w = \mathrm{e}^{\mathrm{i}x}$ such that

$$b\,|n, w\rangle = q^{n+1}w\,|n, w\rangle \tag{33}$$

$$b^*\,|n, w\rangle = q^{n+1}\overline{w}\,|n, w\rangle\,. \tag{34}$$

Now the parameter w looks like a second coherent state label. We try to find a discrete basis of the two-parameter states by expanding the w label in a Fourier series. We introduce the discrete set of orthonormal product states $|n, m\rangle$ which satisfies

$$\langle n_1, m_1|n_2, m_2\rangle = \delta_{n_1n_2}\delta_{m_1m_2} \tag{35}$$

$$a\,|n, m\rangle = \sqrt{[n]}\,|n-1, m\rangle \tag{36}$$

$$a^*\,|n, m\rangle = \sqrt{[n+1]}\,|n+1, m\rangle\,. \tag{37}$$

We first try the Fourier expansion

$$|n, w\rangle = \sum_{m=-\infty}^{\infty} w^m \,|n, m\rangle \tag{38}$$

so that action of b on these basis states is specified as

$$b\,|n, m\rangle = q^{n+1}\,|n, m-1\rangle \tag{39}$$

$$b^*\,|n, m\rangle = q^{n+1}\,|n, m+1\rangle\,. \tag{40}$$

As we can see, the operators b and b^* act as raising and lowering operators on the second state label. The important difference between these states and the old ones is that m is allowed to vary over negative integers as well. Now the main problem is that the definition (38) is convergent only for w on the unit circle. Thus although q was introduced as a convergence parameter we are once more confronted with an expression having convergence problems. To make this expression convergent we will introduce a second deformation parameter p. In this way, we extend the domain of the variable w to the complex plane. Thus we define the new states $|n, w\rangle$ using the product basis $|n, m\rangle$as

$$|n, w\rangle = \sum_{m=-\infty}^{\infty} w^m p^{\frac{m(m-1)}{2}}\,|n, m\rangle \qquad 0 < p < 1. \tag{41}$$

Finally, we can now define the product coherent states $|z, w\rangle$ from the states $|n, w\rangle$ just as we did before,

$$|z, w\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{f_n}}\,|n, w\rangle = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \frac{z^n}{\sqrt{f_n}} w^m p^{\frac{m(m-1)}{2}}\,|n, m\rangle\,. \tag{42}$$

The coherent states are defined for $|z| < 1$ and for all w. The scalar product of the coherent states becomes

$$\langle z_1, w_1|z_2, w_2\rangle = \frac{P(\bar{w}_1 w_2)}{G(\bar{z}_1 z_2)} \qquad \text{where} \quad P(x) = \sum_{m=-\infty}^{\infty} x^m p^{m(m-1)}. \tag{43}$$

We remark that the function $P(x)$ is related to the θ function

$$\theta_3(u, p) = \sum_{n=-\infty}^{\infty} p^{n^2} \mathrm{e}^{2\mathrm{i}nu} \tag{44}$$

by the following relation

$$P(x) = \theta_3\left(\frac{1}{2\mathrm{i}} \ln \frac{x}{p}, p\right). \tag{45}$$

Now the action of a and a* on the coherent states is as before and the action of b and b^* can be evaluated using (39) and (40) as

$$a\,|z, w\rangle = z\,|z, w\rangle \tag{46}$$

$$a^*\,|z, w\rangle = z^{-1}(|z, w\rangle - |q^2 z, w\rangle) \tag{47}$$

$$b\,|z, w\rangle = qw\,|qz, pw\rangle \tag{48}$$

$$b^*\,|z, w\rangle = \frac{qp}{w}\left|qz, \frac{w}{p}\right\rangle. \tag{49}$$

We see now, however, that the additional deformation parameter p appears in these actions which is not specified by the algebra. To correct this we extend the $SU_q(2)$ algebra by introducing two new operators d and d^* such that the action of d on the coherent states is just like a but with the eigenvalue w. Thus d is an operator that becomes a phase operator in the limit $p \to 1$ ($w \to \mathrm{e}^{\mathrm{i}x}$).

$$d\,|z, w\rangle = w\,|z, w\rangle \tag{50}$$

To satisfy this, the action of d and d^* on the basis states $|n, m\rangle$ is defined as

$$d\,|n, m\rangle = p^{1-m}\,|n, m-1\rangle \tag{51}$$

$$d^*\,|n, m\rangle = p^{1-m}\,|n, m+1\rangle\,. \tag{52}$$

Now in addition to the existing $SU_q(2)$ algebra (21)–(24), we have the commutation relations

$$ad = da \tag{53}$$

$$ad^* = d^*a \tag{54}$$

$$db = pbd \tag{55}$$

$$d^*b = pbd^* \tag{56}$$

$$d^*d = p^2dd^* \tag{57}$$

which can be verified using the actions of these operators on the basis states $|n, m\rangle$. So now p appears explicitly in the algebra as well. We can identify p as a second deformation parameter of the algebra after the q-deformation. Thus the coherent states $|z, w\rangle$ are deformed with q in z and with p in w. The operators d and d^* act on the coherent states according to

$$d\,|z, w\rangle = w\,|z, w\rangle \tag{58}$$

$$d^*\,|z, w\rangle = \frac{p^3}{w}\left|z, \frac{w}{p^2}\right\rangle. \tag{59}$$

4. Resolution of unity

Finally, we also write the resolution of unity as an integral over coherent states in the z–w planes as we did before for the coherent states $|z\rangle$. Since w is not restricted to the unit disk and is rather allowed to vary along the entire complex plane we need to generalize the Jackson integral to the interval $(0, \infty)$. This is easily done by defining

$$\int_0^\infty f(x)\,\mathrm{d}_p x = a(1-p)\sum_{k=-\infty}^{\infty} p^k f(ap^k). \tag{60}$$

It is easy to verify that this definition converges to the Riemann integral on the interval $(0, \infty)$ in the limit $p \to 1$. Here a is an arbitrary parameter and can be chosen as 1 as a convention. Now we can write the resolution of unity over circles of radius q^{2n} in the z-plane and over circles of radius p^m in the w-plane as

$$I = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} |n, m\rangle\,\langle n, m| = \int_{|z|<1} \mathrm{d}_{q^2} z \frac{G(q^2\,|z|^2)}{\pi(1-q^2)} \int \mathrm{d}_p w F(|w|^2)\,|z, w\rangle\,\langle z, w| \tag{61}$$

and the weight function $F(|w|^2)$ is given by

$$F(x) = \frac{\sqrt[4]{p}}{\pi(1-p)S(\sqrt[4]{p})} x^{-\frac{\ln x}{4(-\ln p)} - \frac{3}{2}} \tag{62}$$

where

$$S(x) = \sum_{k=-\infty}^{\infty} x^{k^2}. \tag{63}$$

This normalization in (61) is chosen such that one has

$$\langle n_1, m_1| \, I \, |n_2, m_2\rangle = \delta_{n_1 n_2}\delta_{m_1 m_2} = \int_{|z|<1} \mathrm{d}_{q^2} z \frac{G(q^2\,|z|^2)}{\pi(1-q^2)} \times \int \mathrm{d}_p w F(|w|^2)\, \langle n_1, m_1|z, w\rangle \, \langle z, w|n_2, m_2\rangle \, . \tag{64}$$

The normalization of the z-integration is as in (16) and to verify the normalization in the w integration we make use of the formula

$$\int_0^\infty x^m p^{m(m-1)} x^{-\frac{\ln x}{4(-\ln p)}-\frac{3}{2}} \, \mathrm{d}_p x = \frac{(1-p)S(\sqrt[4]{p})}{\sqrt[4]{p}} \tag{65}$$

which can be evaluated using

$$\sum_{k=-\infty}^{\infty} p^k p^{mk} p^{m(m-1)} p^{k\left(\frac{k}{4}-\frac{3}{2}\right)} = p^{m(m-1)} p^{-\frac{(2m-1)^2}{4}} \sum_{k=-\infty}^{\infty} p^{\frac{(k+2m-1)^2}{4}} = \frac{1}{\sqrt[4]{p}} S(\sqrt[4]{p}). \tag{66}$$

5. Conclusions

We have extended the $SU_q(2)$ algebra (21)–(24) by introducing the operators d and d^* (53)–(57) and constructed two-parameter coherent states from the discrete product basis $|n, m\rangle$ such that the coherent states $|z, w\rangle$ are eigenstates of a and d with eigenvalues z and w respectively. In doing so we also introduced a second deformation variable p into the algebra and into the definition of the coherent states. We have also written the resolution of unity as an integral over concentric circles in the z- and w-planes. It is worth remarking that if we look at the commutation relations involving a, a^*, d and d^* only, we find that a and a^* commute with d and d^* and satisfy similar commutation relations among themselves (1), (57). Thus the algebra looks like the direct product of two distinct algebras, where the commutator of a and a^* is deformed with the parameter q and the commutator of d and d^* is deformed with the parameter p. However, the operators b and b^* link these algebras together since b does not commute with either a or d. Furthermore, the parameter q is explicit in the commutation relation between a and b (21) and the parameter p is explicit in the commutation relation between d and b (53).

We would also like to comment that one could replace the relation (57) with one which is more similar to (1) like

$$d^*d - p^2 dd^* = 1 - p^2 \tag{67}$$

keeping the rest of the algebra and introduce another representation using discrete basis states $|n, m, k\rangle$ such that a^* and a act as raising and lowering operators on n, b^* and b act as raising and lowering operators on m, and d and d^* act as raising and lowering operators on k. However, this representation is inconsistent with the extended algebra unless the parameter p is a pure phase and this makes the construction of coherent states nonconvergent in such a representation.

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